Week 7: The Mapping Cylinder II

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Contents

1 Instructions

Please complete all exercises. There are two in total

2 Cofiber Sequences II

Fix a map $f : X \to Y$. The mapping cone $C_f = Y \cup_f CX$ is constructed with a particular universal property in mind. If $g : Y \to Z$ is a map such that the composition gf is null homotopic, then a choice of homotopy $F : gf \simeq *$ gives rise to a map $\underline{g}_F$ as that induced out of the pushout in the following diagram

$$\begin{tikzcd} X \arrow[r, "in_0"] \arrow[d, "f"'] & CX \arrow[d] \arrow[ddr, bend left, "F"] & \\ Y \arrow[r] \arrow[drr, bend right, "g"'] & C_f \arrow[dr, dashed, "\underline{g}_F"] & \\ & & Z. \end{tikzcd} \tag{2.1}$$

We call

$$\underline{g}_F : C_f \to Z, \qquad \begin{cases} y & \mapsto g(y) \\ (x,t) & \mapsto F(x,t) \end{cases} \tag{2.2}$$

the **extension** of g defined by F. It's important to realise that both $\underline{g}_F$ and its homotopy class depend on the choice of homotopy F. In fact we will see some examples below where different choices of F give rise to completely different homotopy classes of extension.

Definition 1 *We say that a three-term sequence of spaces and maps*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{2.3}$$

is a ***homotopy cofiber sequence*** *if i) the composition gf is null homotopic, and ii) there exists a null homotopy $F : g \circ f \simeq *$ for which the extension*

$$\underline{g}_F : C_f \to Z \tag{2.4}$$

is a homotopy equivalence.

We generalise this definition to arbitrarily long sequences of spaces and maps by saying that

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \ldots \to X_n \xrightarrow{f_n} X_{n+1} \to \ldots \tag{2.5}$$

is a ***homotopy cofiber sequence*** *if each three-term subsequence is a homotopy cofiber sequence in the previous sense.* □

Example 2.1

1. The sequence

$$X \xrightarrow{f} Y \to C_f \tag{2.6}$$

   is a homotopy cofiber sequence. The required null homotopy F comes from the pushout which defines C_f. Of course this is the canonical example of such an object, and the definition was formulated to make it so.

2. If $j : A \hookrightarrow X$ is a cofibration, then the 'strict' cofiber sequence

$$A \xrightarrow{j} X \xrightarrow{q} X/A \tag{2.7}$$

   is a homotopy cofiber sequence. The composition is strictly null and in this rather special case we can take F to be the constant homotopy. Then the extension is exactly the homotopy equivalence $C_f \to C_f/CA \cong X/A$ which we used when studying these objects before. □

What is the link between these two examples? Last week in the exercises you figured out how to convert a map $f : X \to Y$ into a cofibration. For ease you worked in the unpointed category and found to replace f with the inclusion $X \hookrightarrow \widetilde{M_j}$ into its unreduced mapping cylinder. The same construction also works in the pointed category when we use the reduced mapping cylinder $M_f = Y \cup X \wedge I_+$. What results is the following strictly commutative diagram of pointed maps

$$\begin{tikzcd} & X \arrow[dl, "j_f"'] \arrow[dr, "f"] & \\ M_f \arrow[rr, "r_f"] & & Y \end{tikzcd} \tag{2.8}$$

in which j_f is a pointed cofibration and r_f is a homotopy equivalence. We get from this the strict cofiber sequence

$$X \xrightarrow{j_f} M_f \xrightarrow{q_f} C_f \tag{2.9}$$

and to go from here to the homotopy cofibration sequence (2.6) is just a matter of using the homotopy equivalence r_f to identify M_f and Y. The need to generalise 'strict' to 'homotopy' is the price we must pay for not starting with a cofibration. In the case that f is a cofibration, then your work last week showed that r_f is a homotopy equivalence *under* X, and the two notions are essentially identical.

Now, we claim that

$$X \xrightarrow{f} Y \xrightarrow{q_f} C_f \xrightarrow{\delta} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{q_{\Sigma f}} C_{\Sigma f} \to \dots \tag{2.10}$$

is a homotopy cofiber sequence. We won't spell out all the details, but what we would like to draw attention to is the disappearance of the annoying minus signs. The map δ is defined by

$$\begin{cases} \delta(y) = * \\ \delta(x,t) = x \wedge t \end{cases} \tag{2.11}$$

and the required null homotopy $F_s : \Sigma f \circ \delta \simeq *$ is given by

$$\begin{cases} F_s(y) = y \wedge s \\ F_s(x,t) = f(x) \wedge ((1-s)t + s) \,. \end{cases} \tag{2.12}$$

Notice that although $\delta q_f = *$ holds strictly we do not use the constant homotopy. Essentially what we've done is absorb the minus sign that appeared in the strict cofiber sequence inside the homotopy F.

Proposition 2.1 *Let $f : X \to Y$ be a map. Then for any space Z the sequence*

$$\begin{gathered} [X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{q^*} [C_f,Z] \xleftarrow{\delta^*} [\Sigma X, Z] \xleftarrow{\Sigma f^*} [\Sigma Y, Z] \xleftarrow{\Sigma q^*} [\Sigma C_f, Z] \longleftarrow \dots \\ \dots \longleftarrow [\Sigma^n X, Z] \xleftarrow{\quad \Sigma^n f^* \quad} [\Sigma^n Y, Z] \xleftarrow{\quad \Sigma^n q^* \quad} [\Sigma^n C_f, Z] \longleftarrow \dots \end{gathered} \tag{2.13}$$

is exact. The first three terms are exact in the sense of pointed sets, the next three terms are exact in the sense of groups, and the remaining terms are exact in the sense of abelian groups.

Proof The proof of exactness of the sequence reduces to proving exactness of each of its three-term subsequences, and exactness here comes from the fact that each three-term subsequence of (2.10) is pointwise equivalent to a strict cofiber sequence. For instance, at the first stage the homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "j_f"] \arrow[d, equal] & M_f \arrow[r] \arrow[d, "r_f", "\simeq"'] & C_f \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "q"] & C_f \end{tikzcd} \tag{2.14}$$

in which the vertical arrows are homotopy equivalences leads to exactness of

$$[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{q_f^*} [C_f, Z]. \tag{2.15}$$

At the next stage we find a suitable null homotopy $F : \delta \circ q_f \simeq *$ and get a homotopy commutative diagram

$$\begin{tikzcd} Y \arrow[r, "q_f"] \arrow[d, equal] & C_f \arrow[r] \arrow[d, equal] & C_{q_f} \arrow[d, "\underline{q}_F", "\simeq"'] \\ Y \arrow[r, "q_f"] & C_f \arrow[r, "\delta"] & \Sigma X. \end{tikzcd} \tag{2.16}$$

We argue as before that applying $[-, Z]$ to the top row of the diagram leads to an exact sequence, so infer from the fact that the vertical arrows in the diagram are homotopy equivalences that

$$[Y, Z] \xleftarrow{q^*} [C_f, Z] \xleftarrow{\delta^*} [\Sigma X, Z] \tag{2.17}$$

is exact.

At the higher stages the homotopy sets obtain group structures from the suspensions. The suspended maps are co-H-maps which induce homomorphisms between the groups. In this case we have to be careful when constructing the coextensions so that these too preserve the group structures. The trick here is to recall that suspension preserves pushouts (cf. *Pointed Homotopy* Co. 1.13), so we have a canonical homeomorphism $\Sigma C_f \cong C_{\Sigma f}$. Now if $F_s : g \circ f \simeq *$ is a null homotopy which induces $\underline{g}_F$, then $\Sigma F_s : \Sigma g \circ \Sigma f$, $x \wedge t \mapsto F_s(x) \wedge t$, is a null homotopy which induces $\underline{\Sigma g}_{\Sigma F} = \Sigma \underline{g}_F$. We leave the task of filling in the details to the reader. ∎

From this point we'll quickly lose the need to differentiate between 'strict' and 'homotopy' cofibrings. We'll generally just refer to them as *cofibration sequences*. While we need the strictness to make the exactness work, the more flexible notion is both easier to work with and more intuitive.

Exercise 2.1 The purpose of this short exercise is to allow you get a handle on how extensions work. Prove that

$$X \to * \to \Sigma X \xrightarrow{=} \Sigma X \to * \to \Sigma^2 X \xrightarrow{=} \Sigma^2 X \to * \to \dots \tag{2.18}$$

a homotopy cofibration sequence. You'll notice that at the beginning of the sequence the obvious null homotopy does *not* generate an extension which meets the second requirement of Definition (1). □

We saw last week how cofibration sequences give rise to long exact sequences in homology and cohomology. We'll leave the reader to formulate a precise statement of the fact that homotopy cofibration sequences also lead to long exact sequences. The following examples give some intuition as to how to use this in practice.

Example 2.2 Suppose that X is obtained from A by attaching an n-cell along a map $\varphi : S^{n-1} \to A$. Then there is a cofiber sequence

$$S^{n-1} \xrightarrow{\varphi} A \to X \xrightarrow{q} S^n \xrightarrow{\Sigma\varphi} \Sigma A \to \Sigma X \to \dots \tag{2.19}$$

The connecting map q in this case is just the quotient map which pinches A to a point. The point is that φ is very unlikely to be an actual cofibration. In the examples below it will be a fibration. Nevertheless we see quite clearly through the exact sequences related to (2.19) how its homotopy class influences the topology of X.

For instance we form the real projective plane by attaching a 2-cell to $S^1 \cong \mathbb{R}P^1$ along the two-sheeted covering projection

$$S^1 \xrightarrow{2} S^1 \to \mathbb{R}P^2 \to S^2 \xrightarrow{2} S^2 \to \Sigma\mathbb{R}P^2 \to \dots \tag{2.20}$$

The cofiber sequences give exact sequences in homology and cohomology as we saw last week and the exactness of

$$\dots \leftarrow H^2S^1 \leftarrow H^2S^1 \leftarrow H^2\mathbb{R}P^2 \leftarrow H^2S^2 \xleftarrow{\times 2} H^2S^2 \leftarrow \dots \tag{2.21}$$

is just the standard computation that $H^2\mathbb{R}P^2 \cong \mathbb{Z}_2$. Another way to interpret this information is that the connected two-sheeted covering projection $S^1 \to S^1$ is the degree 2 map, a fact which takes a little work to prove using degree theory.

For another example, consider the complex projective plane. This is obtained by attaching a 4-cell to $S^2 \cong \mathbb{C}P^1$ along the Hopf map $\eta \in \pi_3 S^2$. This gives a cofiber sequence

$$S^3 \xrightarrow{\eta} S^2 \to \mathbb{C}P^2 \xrightarrow{q} S^4 \xrightarrow{\Sigma\eta} S^3 \to \Sigma\mathbb{C}P^2 \to S^5 \to \dots \tag{2.22}$$

and immediately tells us a couple of things. Firstly we know that the cohomology ring

$$H^*\mathbb{C}P^2 \cong \mathbb{Z}[x]/(x^3) \tag{2.23}$$

is a truncated polynomial ring generated by a degree 2 class x. We infer from this that $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$. But since $\mathbb{C}P^2$ is the homotopy cofiber of η, this implies that $\eta \not\simeq *$. For the mapping cone of the constant map $S^3 \to S^2$ is homotopy equivalent to $S^2 \vee S^4$, so if η were null homotopic, then it would not be possible to have the cofibration sequence (2.22).

Thus we see the presence of a non-trivial element in $\pi_3 S^2$. Eventually we will be able to use so-called *cohomology operations* to show also that $\Sigma^n\mathbb{C}P^2 \not\simeq S^{n+2} \vee S^{n+4}$ for all $n \geq 0$, and at this stage the cofibration sequence (2.22) will also show us the presence of a non-trivial element in $\pi_{n+1}S^n$ for all $n \geq 2$. □

Let us begin to study the naturality of the cofibration sequences. Assume that we have two maps $f : X \to Y$ and $g : Y \to Z$. There are three cofiber sequences we can study, but for the moment we'll only be interested in those corresponding respectively to gf and g

$$X \xrightarrow{gf} Z \to C_{gf} \xrightarrow{\delta} \Sigma X \to \dots \tag{2.24}$$

$$Y \xrightarrow{g} Z \to C_g \xrightarrow{\rho} \Sigma Y \to \dots \tag{2.25}$$

We get a map $\theta : C_{gf} \to C_g$ as that induced by taking pushouts of the rows in the following diagram

$$\begin{tikzcd} Z \arrow[d, equal] & X \arrow[l, "gf"'] \arrow[r] \arrow[d, "f"] & CX \arrow[d, "Cf"] & & C_{gf} \arrow[d, "\theta"] \\ Z & Y \arrow[l, "g"] \arrow[r] & CY & & C_g \end{tikzcd} \tag{2.26}$$

Explicitly

$$\begin{cases} \theta(z) = z \\ \theta(x,t) = (f(x), t). \end{cases} \tag{2.27}$$

We check that this makes the following diagram commute (at least up to homotopy)

$$\begin{tikzcd} X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[r, "\rho"] \arrow[d, equal] & C_{gf} \arrow[r, "\delta"] \arrow[d, "\theta"] & \Sigma X \arrow[r] \arrow[d, "\Sigma f"] & \dots \\ Y \arrow[r, "g"] & Z \arrow[r, "\delta"] & C_g \arrow[r, "\rho"] & \Sigma Y \arrow[r] & \dots \end{tikzcd} \tag{2.28}$$

Here the rows are the cofibration sequences induced by gf and g, respectively, and the connecting maps δ, ρ are defined as in (2.11). Of course by taking suspensions we can make the diagram continuous onwards for as long as we like.

Exercise 2.2 The purpose of this exercise is to greatly strengthen the conclusion of Example 2.2 and show that $\pi_3 S^2$ in fact contains an infinite cyclic summand. We won't be able to conclude using these methods that the group is actualy isomorphic to $\mathbb{Z}$ (it is, and we will show this in the lectures), but it gives us a good chance to get to grips with how some of the techniques developed in this sheet may be used.

1. Fix a space X. Using the definition of homotopy groups from the *Co-H-Spaces* exercises, show that if k is an integer and $\alpha \in \pi_n X$, then

$$k \cdot \alpha = \begin{cases} \alpha + \alpha + \cdots + \alpha \ \ (k \text{ times}) & k > 0 \\ 0 & k = \text{if } 0 \\ -\alpha - \alpha - \cdots - \alpha \ \ (k \text{ times}) & k < 0 \end{cases} \tag{2.29}$$

   is given by the composition

$$k \cdot \alpha : S^n \xrightarrow{\underline{k}} S^n \xrightarrow{\alpha} X \tag{2.30}$$

   where $\underline{k} : S^n \to S^n$ is the degree k map.

2. Now consider the Hopf map $\eta \in \pi_3 S^2$. We will be more definite with its definition in future, but all we need for this exercise is that it is an essential map $S^3 \to S^2$. The argument for this was contained in Example 2.2, so you will need to understand it to proceed. Fix an integer $k \geq 1$ and write

$$C(k) = C_{k \cdot \eta} \tag{2.31}$$

   for the mapping cone of $k \cdot \eta$. Now factor this map as $k \cdot \eta = \eta \circ \underline{k}$ and use the diagram

$$\begin{tikzcd} S^3 \arrow[r, "k\cdot\eta"] \arrow[d, "\underline{k}"] & S^2 \arrow[r] \arrow[d, equal] & C(k) \arrow[r] \arrow[d, "\theta"] & \dots \\ S^3 \arrow[r, "\eta"] & S^2 \arrow[r] & \mathbb{C}P^2 \arrow[r] & \dots \end{tikzcd} \tag{2.32}$$

   to compute the cohomology ring $H^* C(k)$. Here θ is the map induced as in (2.26). You will need to know the action of the degree k map on $H^* S^n$ and the cohomology ring (2.23).

3. Make your conclusions and complete the original claim. $\square$